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A Fractional Cointegration Analysis of Purchasing Power Parity

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A generalized notion of cointegration, called fractional cointegration, is introduced to examine the long-run purchasing power parity (PPP) hypothesis. By allowing deviations from equilibrium to follow a fractionally integrated process, the fractional cointegration analysis can capture a wider range of mean-reversion behavior than standard cointegration analyses. This gain is flexibility in modeling subtle mean-reverting dynamics is found to be important for a proper evaluation of long-run PPP. Empirical results based on historical data for the 1914–1989 period show that PPP reversion exists and can be characterized by a fractionally integrated process in three out of five countries studied. The results support PPP as a long-run phenomenon, though significant short-run deviations from PPP can exist.

KEY WORDS: Fractional cointegration; Fractional integration; Purchasing power parity.

The doctrine of purchasing power parity (PPP) is an important element of international macroeconomics. It is a major building block of standard models of exchange-rate determination—for example, the Frenkel–Bilson and the Dornbusch–Frankel models. It is also sometimes used to provide a benchmark for evaluating the level of an exchange rate in policy discussion.

Although many empirical studies report significant deviations from PPP in the short run, the validity of PPP in the long run remains controversial. For example, Adler and Lehmann (1983), Darby (1983), and Roll (1979) found that the real exchange rate follows closely a random walk, suggesting that shocks have infinitely long-lived effects. In contrast, Abuaf and Jorian (1990) reported evidence supportive of PPP reversion for the 1900–1972 period using multivariate unit-root tests. They studied wholesale price indexes in their analysis. Kim (1990) examined the same data set and also found favorable evidence for long-run PPP using cointegration analysis. Nonetheless, he noted that when consumer price indexes are used, little evidence for long-run PPP can be found. Based on fractional differencing analysis, Diebold, Husted, and Rush (1991) provided evidence of mean reversion in the real exchange rate under the gold standard. For the recent period of float, Huizinga (1987) and Kaminsky (1988) found some evidence of reversion toward PPP using variance ratio tests. Baillie and Selover (1987), Corbae and Ouliaris (1988), Mark (1990), and Taylor (1988), however, failed to find cointegration between nominal exchange rates and relative prices, implying that the two series tend to drift apart without bound. The results of impulse-response analysis by Mark (1990) also showed little support for long-run PPP.

The apparently mixed findings reflect a major problem associated with testing of long-run PPP. A test for long-run PPP entails proper modeling of the low-frequency dynamics of economic variables and their equilibrium relationship, while allowing for significant deviations from equilibrium in the short run. Empirical results can crucially depend on the power of the statistical technique employed to separate the low-frequency from the high-frequency dynamics. The use of a statistical procedure that can identify a rich class of low-frequency dynamics and detect long-run relationships from noisy data thus appears desirable.

This study examines the relevance of long-run PPP using a fractional cointegration approach that integrates the notions of cointegration, suggested by Engle and Granger (1987), and of fractional differencing, introduced by Granger and Joyeux (1980) and Hosking (1981) to economics. The concept of fractional cointegration was mentioned by Granger (1986), but its practical relevance has not yet been shown in any empirical work. Granger (1987) discussed the dominant property of various generalized time series processes, including fractionally integrated processes. For a fractionally integrated series, the dominant property is determined by its integration order relative to that of other series. According to Granger (1987), two series having the same dominant property are said to be codominated if a linear combination of the two series does not have the dominant property. The fractional cointegration concept entertained in this article belongs to the class of generalized cointegration or codomination examined by Granger (1987).

In general, the cointegration analysis enables us to test for a long-run relationship, with little restriction on the short-run dynamics. Tests for cointegration often draw on unit-root tests that presume the order of integration of the equilibrium error to be an integer. A system of economic variables, however, can be fractionally cointegrated such that its equilibrium errors follow a fractionally integrated process (Granger 1986). The notion of fractional cointegration is of economic relevance in that it implies the existence of a long-run equilibrium relationship, since fractionally integrated equilibrium errors can be shown to be mean-reverting, though they exhibit significant persistence in the short run. By avoiding the knife-edged unit-root/no-unit-root distinction in the equilibrium error, the fractional cointegration analysis permits a wider range of mean-reversion behavior than standard cointegration analyses. This gain in flexibility in modeling subtle mean-reverting dynamics is shown to be important for a proper evaluation of long-run PPP.

In related research, Diebold et al. (1991) applied tests for fractional integration to detect reversion toward PPP under the gold standard. Their analysis was conducted in a univariate setting based on real exchange-rate data, which extend back from 1913 to the beginning of the nineteenth century. In contrast, our fractional cointegration approach enables estimation of the long-run PPP relationship in a multivariate framework without imposing a priori any restriction such as the homogeneity condition on the short- or long-run dynamics. In this regard, our methodology is relatively general, which gives it potentially wider applications in other areas. Moreover, we examine a different historical data set for the 1914–1989 period. In contrast to the gold-standard data, our data encompass different periods of exchange-rate systems and periods of great economic instability and turbulence, including two world wars and two oil crises. In this way, our results complement those of Diebold et al. (1991).

This article is organized as follows. Section 1 briefly discusses the PPP relationship. Section 2 introduces the notions of fractional integration and cointegration. Section 3 outlines the cointegration tests for long-run PPP and reports empirical results. Section 4 concludes.

1. THE PPP RELATIONSHIP

The PPP doctrine suggests that currencies are valued for the goods they can buy and, in equilibrium, a given basket of goods should cost the same at home and aboard in the presence of international arbitrage. This implies a long-run equilibrium relationship between national price levels expressed in common currency units. For the purpose of empirical testing, the PPP relationship is written as

$$sp_t = \alpha_0 + \alpha_1 p_t + e_t, \tag{1}$$

where α_0 is some constant, sp_t is the foreign price index converted to domestic currency units, p_t is the domestic price index, and e_t is an error term capturing deviations from PPP. All variables are in logarithms. A similar PPP specification has been considered by, for example, Frenkel (1981). Under the homogeneity condition, α_1 is equal to unity. Taylor (1988) noted that, since observed price indexes are imperfect proxies at best for the theoretical price variables, the homogeneity condition does not necessarily hold empirically. Hence α_1 should be estimated rather than imposed a priori to be unity. A necessary condition for PPP to hold in the long run is that e_t is a mean-reverting process; that is, the effect of a shock to the PPP relationship will die out. This forms the basis for a cointegration test of long-run PPP. If sp_t and p_t are found to be cointegrated, deviations from a linear combination of the variables will be mean-reverting, implying that sp_t and p_t are tied together in the long run. The cointegration approach is useful, since it allows data to determine the underlying long-run relationship and its short-run deviations without imposing the homogeneity condition. In general, the cointegration approach can be applied whether the homogeneity condition holds or not.

An issue concerns which aggregate price index should be used for PPP calculations, as discussed by Frenkel (1978). Commonly the choice is between wholesale price indexes (WPI's) and consumer price indexes (CPI's). WPI's place a heavier weight on tradables than CPI's, and WPI's tend to yield more favorable test results to long-run PPP than CPI's (e.g., Kim 1990). The use of WPI's is, however, sometimes under the criticism that the relationship between traded goods prices and exchange rates comes close to a truism. To be important, PPP is a macroeconomic concept that should be measured by broadly based indexes such as CPI's.

The data examined in this article are annual data for the period 1914–1972 taken from Lee (1978) that are extended to 1989 based on the International Monetary Fund's International Financial Statistics data tape. The price levels, p_t and sp_t, are measured by CPI's. Five bilateral intercountry relations are considered between the United States as the home country and the United Kingdom, France, Italy, Canada, and Japan as the foreign countries. Foreign price levels are expressed in terms of U.S. currency units by multiplying the price indexes with the relevant exchange rates.

2. FRACTIONAL INTEGRATION AND COINTEGRATION

The use of the notion of cointegration as a long-run equilibrium relationship between time series was suggested by Engle and Granger (1987) and Granger (1986). A series is said to be integrated of order d, denoted by $I(d)$, if it has a stationary, invertible autoregressive moving average (ARMA) representation after applying the differencing operator $(1 - L)^d$. When d is not an integer, the series is said to be fractionally integrated.

Consider in general a pair of series, x_{1t} and x_{2t}, which are I(d). Let $X_t = (x_{1t}, x_{2t})'$. The linear combination

$$z_t = \alpha X_t \tag{2}$$

will generally also be I(d). If a vector α exists such that z_t is I($d - b$) with $b > 0$, however, x_{1t} and x_{2t} are said to be cointegrated of order (d, b), and $\alpha X_t = 0$ represents an equilibrium constraint operating on the long-run components of X_t. Moreover, the cointegrated system has an error correction representation of the form

$$H(L)(1 - L)^d X_t = -\tau[(1 - (1 - L)^b)](1 - L)^{d-b} z_t + c(L)\varepsilon_t, \tag{3}$$

where $H(L)$ is a matrix polynomial in the lag operator L with $H(0) = \mathrm{I}$, $c(L)$ is a lag polynomial with $c(1)$ finite, and ε_t is a white-noise disturbance term (Granger 1986). Note that the lag function $[(1 - (1 - L)^b)]$, if expanded in powers of L, has no term in L^0 and so only lagged z_t enters into the right side of Equation (3).

The typical case considered in empirical work is one in which $b = d = 1$; that is, x_{1t} and x_{2t} are I(1) and z_t is I(0). Cointegration thus requires that the equilibrium error, z_t, is mean-reverting, even though x_{1t} and x_{2t} wander widely. The mean reversion behavior of the equilibrium error is of key interest, if economic theory suggests a long-run equilibrium relationship between x_{1t} and x_{2t}. Unless the equilibrium error exhibits mean reversion, a shock to the system will tend to drive x_{1t} and x_{2t} out of equilibrium permanently, making the notion of equilibrium have little relevance even in the long run. A testing procedure due to Engle and Granger (1987) is widely used to test for cointegration for its intuitive economic interpretations. The Engle–Granger procedure consists of two steps: Regress x_{1t} on x_{2t} (or x_{2t} on x_{1t}) as the equilibrium or cointegrating regression, and then check if its residual is I(0) or not using a unit-root test. If the residual is found to be I(0), the null hypothesis of no cointegration is rejected.

Analytically, however, the strict I(1) and I(0) distinction is arbitrary. This point is important because, for the equilibrium error to be mean-reverting, it does not have to be I(0) exactly. Fractionally integrated processes, as discussed by Granger and Joyeux (1980) and Hosking (1981), also display mean reversion.

A fractionally integrated process can be represented by

$$C(L)(1 - L)^d z_t = D(L)v_t, \tag{4}$$

where $C(L) = 1 - C_1L - \cdots - C_pL^p$, $D(L) = 1 + D_1L + \cdots + D_qL^q$, all roots of $C(L)$ and $D(L)$ lie outside the unit circle, v_t is iid $(0, \sigma^2)$, and the fractional differencing operator

$$(1 - L)^d = \sum_{k=0}^{\infty} \Gamma(k - d)L^k/\{\Gamma(k + 1)\Gamma(-d)\}, \tag{5}$$

where $\Gamma(\cdot)$ is the gamma function. Model (4) is referred to as the autoregressive fractionally integrated moving average (ARFIMA) model, and it extends the standard (autoregressive integrated moving average) ARIMA (p, d, q) model to real values of d. For $0 < d < .5$, the autocorrelations of z_t show a hyperbolic decay at a rate proportional to k^{2d-1}, in contrast to a faster, geometric decay of a stationary ARMA process (Hosking 1981). Due to the presence of such significant dependence between distant observations, the ARFIMA process is often called a long-memory process. The long-memory behavior of z_t can also be seen from its spectral density, $f_z(\omega)$, which behaves like ω^{-2d} as $\omega \to 0$. For $d > 0$, $f_z(\omega)$ is unbounded at frequency $\omega = 0$ rather than bounded as for a stationary ARMA series. An appeal of the ARFIMA model is its ability to capture a wide variety of low-frequency behavior with a single parameter, d.

Model (4) includes I(1); that is, $d = 1$, as a special case. The distinction between $d = 1$ and $d < 1$ is crucial in terms of the mean-reversion property of z_t, and so is the cointegration property of x_{1t} and x_{2t}. Although the effect of any shock is known to persist forever for an I(1) process, it dies out, albeit slowly, for an I(d) process with $d < 1$. This can be seen by studying the moving average representation for $(1 - L)z_t$:

$$(1 - L)z_t = A(L)v_t, \tag{6}$$

where $A(L) = 1 + A_1L + A_2L^2 + \cdots$, derived from

$$A(L) = (1 - L)^{1-d}\Phi(L) \tag{7}$$

with $\Phi(L) = C^{-1}(L)D(L)$. The moving average coefficients A_i's are called the impulse responses (Campbell and Mankiw 1987). The impact of a unit innovation at time t on the value of z at time $t + k$ is equal to $1 + A_1 + A_2 + \cdots + A_k$. For a mean-reverting process, the infinite cumulative impulse response $A(1)$ equals 0, implying no long-run impact of the innovation on the value of z. Using Equation (5) to find the series representation for $(1 - L)^{1-d}$, Equation (7) can be written as

$$A(L) = F(d - 1, 1, 1; L)\Phi(L), \tag{8}$$

where $F(\cdot)$ is the hypergeometric function defined by

$$F(m, n, p; L) = \frac{\sum_{j=0}^{\infty} \Gamma(m + j)\Gamma(n + j)\Gamma(p)L^j}{\{\Gamma(m)\Gamma(n)\Gamma(p + j)\Gamma(j + 1)\}}. \tag{9}$$

Using some known properties of the hypergeometric function (Gradshteyn and Ryzhik 1980, pp. 1039–1042), it can be shown that $F(d - 1, 1, 1; 1) = 0$ for $d < 1$. It follows that

$$A(1) = F(d - 1, 1, 1; 1)\Phi(1) = 0 \tag{10}$$

for $d < 1$. Hence an I(d) process with d less than unity is mean-reverting. The mean-reverting property can also be seen in the frequency domain by observing that $(1 - L)z_t$ has an integration order less than 0, which is known to imply a zero spectrum at the origin.

Note that when $.5 \le d < 1$, the z_t process is covariance nonstationary because its variance is not finite (Hosking 1981). Nonetheless, the z_t process is mean-reverting, since an innovation has no permanent effect on the value of z_t. This is in contrast to an I(1) process, which is both covariance nonstationary and not mean-reverting. For an I(1) process, the effect of an innovation can persist forever.

To the extent that the equilibrium error can display slow mean reversion, not captured by usual I(0) processes, a general test for cointegration should allow for fractional cointegration. Although Engle and Granger (1987) and Granger (1986) noted that the notion of cointegration can well apply to fractionally integrated processes, no empirical work to date has been done on fractional cointegration, and its practical relevance is yet to be established.

In this article, the long-run PPP hypothesis is examined using fractional cointegration analysis. For this application we will be mainly concerned with the case in which x_{1t} and x_{2t} are I(1), since the I(1) hypothesis cannot be rejected statistically for the individual time series that are to be examined. In this case, if the equilibrium error z_t can be found to be $I(1 - b)$ with $b > 0$, though not necessarily I(0), x_{1t} and x_{2t} are fractionally cointegrated and the effect of a shock to the system will eventually die out so that an equilibrium relationship between x_{1t} and x_{2t} will prevail in the long run.

3. TESTING FOR LONG-RUN PPP

As noted in Section 1, PPP suggests a long-run equilibrium relationship between a pair of variables, namely sp_t and p_t. Following the cointegration analysis, a test for this relationship can be performed by testing their equilibrium error, e_t, for fractional integration, and the Engle–Granger (1987) procedure readily lends itself to such analysis. The test proceeds as follows: Estimate Equation (1) as the cointegrating regression and examine if its least squares residual is $I(d)$ with $d < 1$ or not. The latter requires direct estimation of the integration parameter d, without the knife-edged I(1) and I(0) distinction typically maintained in unit-root tests.

For standard cointegration analyses, in which cointegration of order (1, 1) is considered, Stock (1987) showed that the least squares estimate of the cointegrating parameter is consistent and converges in probability at the rate of $O(T)$ rather than the usual rate of $O(T^{1/2})$. In the fractional cointegration analysis here, the least squares estimate is also consistent, though with possibly different convergence rates according to the actual order of cointegration. Specifically, under the general hypothesis of cointegration of order (d, b) with $b > 0$, it is shown later that the least squares estimate is consistent and converges at the rate of $O(T^b)$. This result includes Stock's (1987) convergence result as a special case in which $b = 1$.

3.1 Least Squares Estimation Under Fractional Cointegration

Consider two time series, x_t and y_t, which are $I(d)$ and are fractionally cointegrated of order (d, b) such that there exists a ζ that

$$y_t = \zeta x_t + \varepsilon_t, \tag{11}$$

where ε_t is $I(d - b)$ with $d > \frac{1}{2}$ and $d \geq b > 0$. The least squares estimator of ζ is given by

$$\hat{\zeta} = \sum_{t=1}^{T} x_t y_t \Big/ \sum_{t=1}^{T} x_t^2 = \zeta + \sum_{t=1}^{T} x_t \varepsilon_t \Big/ \sum_{t=1}^{T} x_t^2. \tag{12}$$

The convergence rate of $\hat{\zeta}$ thus depends on those of $\Sigma_{t=1}^{T} x_t \varepsilon_t$ and $\Sigma_{t=1}^{T} x_t^2$. This is examined in two possible situations:

Case 1 ($d - b > \frac{1}{2}$). By the Cauchy–Schwarz inequality, we have

$$\sum_{t=1}^{T} x_t^2 \sum_{t=1}^{T} \varepsilon_t^2 \geq \left(\sum_{t=1}^{T} x_t \varepsilon_t\right)^2. \tag{13}$$

This implies that

$$\left(\sum_{t=1}^{T} x_t^2/T^{2d}\right)\left(\sum_{t=1}^{T} \varepsilon_t^2/T^{2(d-b)}\right) \geq \left(\sum_{t=1}^{T} x_t \varepsilon_t/T^{2d-b}\right)^2. \tag{14}$$

Since it is known that

$$\sum_{t=1}^{T} x_t^2 = O(T^{2d}) \quad \text{and} \quad \sum_{t=1}^{T} \varepsilon_t^2 = O(T^{2(d-b)}), \tag{15}$$

Equation (14) implies that $\Sigma_{t=1}^{T} x_t \varepsilon_t = O(T^\tau)$ with $\tau \leq 2d - b$. In other words, $\Sigma_{t=1}^{T} x_t^2/T^{2d}$ is bounded, and $\Sigma_{t=1}^{T} x_t \varepsilon_t/T^{2b-d+\delta}$ converges in probability to 0 for all $\delta > 0$. It then follows from Equation (12) that

$$T^{b-\delta}(\hat{\zeta} - \zeta) = \left(\sum_{t=1}^{T} x_t \varepsilon_t/T^{2d-b+\delta}\right) \Big/ \left(\sum_{t=1}^{T} x_t^2/T^{2d}\right) \tag{16}$$

converges in probability to 0 for all $\delta > 0$.

Case 2 ($\frac{1}{2} > d - b \geq 0$). In this case

$$\sum_{t=1}^{T} x_t^2 = O(T^{2d}) \quad \text{and} \quad \sum_{t=1}^{T} \varepsilon_t^2 = O(T), \tag{17}$$

since ε_t is $I(d - b)$, which is a stationary process for $d - b < \frac{1}{2}$. Applying the Cauchy–Schwarz inequality, (13) yields

$$\left(\sum_{t=1}^{T} x_t^2/T^{2d}\right)\left(\sum_{t=1}^{T} \varepsilon_t^2/T\right) \geq \left(\sum_{t=1}^{T} x_t \varepsilon_t/T^{d+1/2}\right)^2. \tag{18}$$

This suggests that $\Sigma_{t=1}^{T} x_t \varepsilon_t = O(T^\tau)$ with $\tau \leq d + \frac{1}{2}$. A tighter bound, however, can be obtained by observing that

$$\sum_{t=1}^{T} x_t \varepsilon_t/T^{2d-b} = \sum_{t=1}^{T} (x_t/T^{d-1/2})(\varepsilon_t/T^{d-b-1/2})/T \tag{19}$$

converges in distribution to some function of Brownian motions, following from the functional central limit theorem (Herrndorf 1984, p. 142). This implies that $\Sigma_{t=1}^{T} x_t \varepsilon_t = O(T^\tau)$ with $\tau \leq 2d - b$, so the result in (16) still holds here.

3.2 A Test for Fractional Cointegration

The hypothesis of fractional cointegration raises the problem of testing for fractional integration. Diebold and Rudebusch (1991) and Sowell (1990a) observed that standard unit-root tests such as the Dickey–Fuller test may have low power against fractional alternatives. In this article, a spectral regression-based test due to Geweke and Porter-Hudak (1983) is used to detect fractional integration in the equilibrium error e_t, and its test performance is investigated in Monte Carlo experiments. The Geweke–Porter-Hudak (GPH) test provides a general test for fractional integration that is not dependent on nuisance parameters of the underlying process—namely, the parameterization of the ARMA part of the process. The GPH test makes use of the fact that the spectral density of $G_t = (1 - L)e_t$ is given by

$$\begin{aligned} f_G(\omega) &= |1 - \exp(-i\,\omega)|^{-2(d-1)} f_u(\omega) \\ &= (2\sin(\omega/2))^{-2(d-1)} f_u(\omega), \end{aligned} \tag{20}$$

where $u_t = \Phi(L)v_t$ is a stationary process and $f_u(\omega)$ is its spectral density. Consider a sample series of G_t of size T. Taking logarithms of (20) and evaluating at harmonic frequencies $\omega_j = 2\pi j/T$ ($j = 0, \ldots, T - 1$), we have

$$\begin{aligned} \ln(f_G(\omega_j)) = \ln(f_u(0)) - (d - 1)\ln(4\sin^2(\omega_j/2)) \\ + \ln(f_u(\omega_j)/f_u(0)). \end{aligned} \tag{21}$$

For low-frequency ordinates ω_j near 0, say $j \le n << T$, the last term is negligible compared with the other terms. Adding $\mathrm{I}(\omega_j)$, the periodogram at ordinate j, to both sides of (21) yields

$$\begin{aligned} \ln(\mathrm{I}(\omega_j)) = \ln(f_u(0)) - (d - 1)\ln(4\sin^2(\omega_j/2)) \\ + \ln(\mathrm{I}(\omega_j)/f_G(\omega_j)). \end{aligned} \tag{22}$$

This suggests estimating d using a simple linear regression equation

$$\begin{aligned} \ln(\mathrm{I}(\omega_j)) = \beta_0 + \beta_1 \ln(4\sin^2(\omega_j/2)) + \varepsilon_t, \\ j = 1, 2, \ldots, n, \end{aligned} \tag{23}$$

where ε_t, equal to $\ln(\mathrm{I}(\omega_j)/f_G(\omega_j))$, is asymptotically iid across harmonic frequencies and $n = g(T)$ is an increasing function of T. The theoretical asymptotic variance of ε_t is known to be equal to $\pi^2/6$, which is often imposed in estimation to raise efficiency. Under some regularity conditions on $g(T)$, satisfied by, for example, T^μ for $0 < \mu < 1$, Geweke and Porter-Hudak (1983) showed that the least squares estimate of β_1 provides a consistent estimate of $1 - d$ and hypothesis testing concerning the value of d can be based on the t statistic of the regression coefficient.

In testing for fractional cointegration, however, the critical values for the GPH test are nonstandard, and those derived from the standard distribution cannot be used directly to evaluate the GPH estimate of d. This is because e_t is not actually observed but obtained from minimizing the residual variance of the cointegration regression, and the residual series thus obtained tends to bias toward being stationary. The null hypothesis of no cointegration is therefore expected to be rejected more often than suggested by the nominal size of the GPH test. A similar problem in testing for cointegration using unit-root tests was discussed by Engle and Granger (1987). To cope with the problem, the empirical size of the GPH test in finite samples can be obtained using the simulation approach.

For comparison, a widely used cointegration test is also applied. Specifically, the augmented Dickey–Fuller (ADF) unit-root test, recommended by Engle and Granger (1987), is considered. The ADF test statistic is given by the usual t statistic for b_0 in

$$\begin{aligned} (1 - L)e_t = b_0 e_{t-1} + b_1(1 - L)e_{t-1} \\ + \cdots + b_p(1 - L)e_{t-p} + \xi_t, \end{aligned} \tag{24}$$

where the lag parameter p can be selected using some model-selection procedure such as the Akaike information criterion (AIC) and the Schwarz information criterion (SIC).

Cointegration is sometimes tested by estimating an error-correction model (e.g., Edison 1987). In the fractional cointegration framework, the error-correction model is given by Equation (3). Efficient estimation of this general error-correction model appears not straightforward. Hence this approach is not pursued here.

3.3 The Size and Power of the GPH Test for Cointegration

The empirical size of the GPH test corresponding to our sample size ($T = 76$) is obtained using the Monte Carlo method in 50,000 replications, assuming the true system is of two I(1) processes that are not cointegrated. The choice of the number of low-frequency ordinates, n, used in the GPH spectral regression necessarily involves judgment. Although a too large value of n will cause contamination of the d estimate due to medium- or high-frequency components, a too small value of n will lead to imprecise estimates due to limited degrees of freedom in estimation. Hence a range of values of μ was used for the sample-size function $n = T^\mu$. In view of the recommended choice by Geweke and Porter-Hudak (1983) based on forecasting experiments, we report results for $\mu = .55$, .575, and .60. This set of choice of μ also yielded good test performance in our simulation experiment. Table 1 contains the empirical size of the GPH test for cointegration. We observe that the empirical distribution is negatively skewed. This is consistent with our earlier discussion that the use of standard critical values will make the cointegration test reject the null hypothesis too often.

The power property of the GPH test relative to the ADF test is also examined using the Monte Carlo method. Both tests are performed against autoregressive alter-

Table 1. The Empirical Size of the GPH Test for Cointegration

Percentile	μ = .55	μ = .575	μ = .60
.5%	−3.299	−3.288	−3.230
1.0%	−2.886	−2.900	−2.879
2.5%	−2.359	−2.363	−2.360
5.0%	−1.954	−1.955	−1.964
10.0%	−1.515	−1.520	−1.531
20.0%	−1.029	−1.031	−1.027
30.0%	−.693	−.690	−.693
40.0%	−.415	−.416	−.422
50.0%	−.167	−.168	−.166
60.0%	.077	.078	.082
70.0%	.341	.338	.345
80.0%	.634	.641	.637
90.0%	1.039	1.044	1.045
95.0%	1.380	1.378	1.379
97.5%	1.691	1.695	1.681
99.0%	2.047	2.044	2.029
99.5%	2.293	2.289	2.262
Mean	−.211	−.211	−.212
Skewness	−.320	−.320	−.318
Kurtosis	3.639	3.613	3.543

NOTE: The sample size for the GPH spectral regression is given by $n = T^{\mu}$, where T (= 76) is the number of available observations. The empirical size is obtained based on 50,000 replications in simulation, assuming that the true system is of two noncointegrated random-walk processes.

natives and fractional alternatives. The Monte Carlo experiment conducted is described in the Appendix, and the power estimates are given in Tables 2 and 3. The power of a test is measured as the percentage of the time the test can reject a false null hypothesis of no cointegration. Given that the sample size under consideration is not large, a 10% significance level, together with the standard 5% level, is used in the cointegration tests. In general, the GPH test for cointegration appears to have slightly better statistical power against autoregressive alternatives than the ADF test. When testing against fractional alternatives, the GPH test performs even better relative to the ADF test. This is particularly the case when the fractional integration parameter lies between .35 and .65. In other words, the difference in power between the ADF test and the GPH test is relatively pronounced when the true process is of an integration order lying between 0 and 1, but not close to either. This result reflects the fact that the ADF test assumes a strict I(0) and I(1) distinction, whereas the GPH test does not. In general, the simulation results illustrate that, although the GPH test performs at least as well as the ADF test against the usual autoregressive alternatives, there is potentially a significant power advantage of the GPH test over the ADF test against the fractional alternatives. A more systematic study of the relative power of the GPH test and the ADF test in testing for cointegration is conducted elsewhere. The results indicate that, although the power of either test rises as the sample size increases, the potential power advantage of the GPH test over the ADF test is particularly relevant for a sample size of 200 or less. In addition, the GPH test is more powerful than the ADF test in testing against ARFIMA alternatives. The performance of the two tests is next examined when applied to tests for long-run PPP using actual data.

3.4 Results of Cointegration Tests

The price series are each first checked for a unit root using the ADF test, which allows for a drift and a trend. The GPH test is also performed to check for fractional integration in the individual series. The unit-root hypothesis can be tested by determining whether or not the GPH estimate of d from the first-differenced series significantly differs from 0. The results are reported in Table 4 (p. 111). The ADF test statistics indicate that for all of the series examined the hypothesis of a unit root cannot be rejected even at the 10% level of significance. The GPH test statistics confirm the results and cannot reject the I(1) hypothesis. The results in general support the hypothesis that the price series are integrated of order 1.

Table 5 contains the results of the ADF test for cointegration. The ADF statistics are obtained from Regression (24) with the lag parameter p selected using both the AIC and the SIC. As can be seen in Table 5, the statistical results are apparently mixed. In the cases of France, Italy, and Canada, the hypothesis of no cointegration can be rejected at the 10% level or better, supporting that the series sp_t and p_t are cointegrated,

Table 2. The Power of the ADF and GPH Tests for Cointegration, Against Autoregressive Alternatives: $(1 - \phi L)u_{2t} = \varepsilon_{2t}$

		ϕ									
Size	Test	.95	.85	.75	.65	.55	.45	.35	.25	.15	.05
5%	ADF (p = 4)	.07	.16	.29	.42	.53	.61	.66	.73	.75	.77
	GPH (μ = .55)	.07	.17	.33	.49	.59	.64	.67	.69	.68	.66
	GPH (μ = .575)	.07	.17	.35	.52	.63	.69	.73	.75	.74	.72
	GPH (μ = .60)	.07	.18	.37	.56	.70	.76	.81	.84	.83	.83
10%	ADF (p = 4)	.14	.28	.46	.60	.71	.78	.82	.86	.88	.89
	GPH (μ = .55)	.14	.29	.50	.66	.75	.78	.81	.82	.81	.79
	GPH (μ = .575)	.14	.30	.52	.69	.78	.82	.85	.86	.85	.84
	GPH (μ = .60)	.14	.30	.54	.72	.82	.87	.90	.91	.92	.91

NOTE: Estimates of the power of each test are based on 10,000 replications. The Monte Carlo experiment conducted is described in the Appendix.

Table 3. The Power of the ADF and GPH Test for Cointegration, Against Fractional Alternatives: $(1 - L)^d u_{2t} = \varepsilon_{2t}$

Size	Test	d .95	.85	.75	.65	.55	.45	.35	.25	.15	.05
5%	ADF (p = 4)	.06	.07	.10	.14	.19	.26	.36	.50	.61	.73
	GPH (μ = .55)	.06	.09	.15	.21	.30	.37	.47	.56	.61	.64
	GPH (μ = .575)	.06	.10	.16	.24	.33	.42	.53	.62	.67	.71
	GPH (μ = .60)	.06	.11	.18	.28	.40	.52	.63	.73	.78	.81
10%	ADF (p = 4)	.11	.13	.18	.24	.32	.41	.53	.67	.78	.87
	DPH (μ = .55)	.12	.17	.26	.35	.46	.56	.65	.72	.76	.78
	GPH (μ = .575)	.12	.18	.27	.38	.50	.60	.71	.77	.81	.83
	GPH (μ = .60)	.12	.19	.30	.43	.57	.68	.79	.85	.88	.90

NOTE: See Note to Table 2.

as suggested by long-run PPP. In the cases of the United Kingdom and Japan, however, little evidence of cointegration between sp_t and p_t is found. Reverse cointegrating regressions of p_t on sp_t were also performed. The results were found to differ very little between normalizing the cointegration regressions on the foreign prices and on the domestic prices.

The likelihood ratio test recently suggested by Johansen (1991) was also applied to test for cointegration in a vector autoregression framework for (sp_t, p_t). Critical values for the test were tabulated by Johansen and Juselius (1990a). The Johansen test yielded results qualitatively similar to the ADF test. In addition, the homogeneity condition was formally tested. The condition is rejected in the case of Japan at the 5% level, although it appears to hold in other cases.

The GPH test for cointegration is next performed, and its results are reported in Table 6. The results vary little across the different values of μ under consideration. Table 6 shows that all of the estimates of d lie between 0 and 1, suggesting possible fractional integration behavior. Moreover, results of formal hypothesis testing indicate significant evidence of $d < 1$ in all cases, though the evidence for Canada seems relatively marginal. Moreover, in all cases but one (Italy), the estimates of d are significantly greater than 0. The results on the whole indicate the presence of cointegration and possibly fractional cointegration between sp_t and p_t. The GPH test results thus provide a wider and more significant support for long-run PPP than the ADF test results.

The GPH test results are particularly interesting when compared with those based on the ADF test for cointegration in individual cases. In the cases of the United Kingdom and Japan, in which the ADF test fails to find cointegration between sp_t and p_t, the GPH test finds evidence of fractional cointegration between the two series. We observe that all of the GPH estimates of d for the United Kingdom and Japan are within the range of .35 and .65. According to our results on power simulation reported earlier, for $.35 < d < .65$, the GPH test tends to be relatively more powerful than the ADF test in finding cointegration. Hence the presence of fractional cointegration may account for the seemingly negative results obtained from the ADF test in these two cases.

In an earlier version of this article (Cheung and Lai 1990) the mean-reverting behavior of the equilibrium error e_t was further examined using the exact maximum likelihood method discussed by Li and McLeod (1986) and Sowell (1990b). We found qualitatively the same results as those from the GPH procedure—the d estimates were significantly less than unity in all cases. Indeed, the maximum likelihood estimates were in some

Table 4. Results of Unit-Root Tests on Individual Price Series

Country	Lag p (AIC)	ADF	Lag p (SIC)	ADF	GPH μ = .55	μ = .575	μ = .60
Canada	7	−1.831	7	−1.831	1.251	1.400	1.046
France	4	−2.927	4	−2.927	−1.246	−1.020	−1.265
Italy	4	−1.897	4	−1.897	−1.088	−1.233	−.759
Japan	4	−1.082	4	−1.082	2.152*	1.523	.827
U.K.	4	−1.148	2	−1.001	.013	.143	−.130
U.S.	4	−1.429	4	−1.429	1.095	1.361	1.158

NOTE: All price variables are expressed in terms of U.S. dollars and are in logarithms. The lag parameter p of the ADF test is selected using two information criteria, the AIC and the SIC, based on a maximum lag of 8. The sample size for the GPH spectral regression is given by $n = T^{\mu}$. The GPH test statistics are the t statisics from the spectral regressions, constructed by imposing the known theoretical error variance of $\pi^2/6$. For the GPH test, the hypothesis of $d = 1$ is tested against the alternative of $d \neq 1$. Critical values for the ADF test are based on Fuller (1976, p. 373); they are given by −3.15 (10%) and −3.45 (5%). Significance is indicated by * at the 10% level or ** at the 5% level.

Table 5. Results of the ADF Test for Cointegration

Country	Lag p (AIC)	ADF	Lag p (SIC)	ADF
Canada	7	−3.118*	7	−3.118*
France	4	−3.701**	4	−3.701**
Italy	4	−2.981*	4	−2.981*
Japan	4	−2.304	4	−2.304
U.K.	4	−2.781	4	−2.781

NOTE: Critical values for the ADF test were tabulated by Engle and Yoo (1987, p. 158); they are given by −2.91 (10%) and −3.17 (5%) for $T = 100$. Using simulations, we obtain that the critical values are given by −2.923 (10%) and −3.217 (5%) for our sample size $T = 76$. Significance is indicated by * at the 10% level or ** at the 5% level.

cases much below the GPH estimates. This can be explained by the property that the maximum likelihood procedure tends to underestimate the true d estimate, as reported by Cheung and Diebold (1991) based on Monte Carlo analysis. Given the potential bias of the maximum likelihood estimates, the corresponding results are not reported here.

Note that the long sample period involves data from different exchange-rate regimes. In general, flexible exchange-rate periods tend to be associated with higher variabilities in exchange rate and price changes than fixed exchange-rate periods. Hence the variances of our data series may not be stationary across exchange-rate regimes. This kind of nonstationarity will not pose any problem to the present analysis, nonetheless. Monte Carlo results reported by Cheung (1991) suggest that the GPH test is robust to variance shifts and conditional heteroscedastic effects.

Subsample analysis was also conducted based on three subsample periods, 1914–1946, 1947–1972, and 1973–1989. These subperiods correspond roughly to the pre-Bretton Woods, the Bretton Woods, and the post-Bretton Woods periods. The statistical results indicated that in no case could the hypothesis of no cointegration be rejected in any of the subsample periods. Such negative results can be due to the small number of degrees of freedom and the short time span of the data considered in subsample analysis. Hakkio and Rush (1991) recently illustrated the desirability of using long sample periods for testing for cointegration. The use of a longer time span of data can provide more information about the low-frequency dynamics that are important for detecting the long-run property of cointegration. Our empirical results apparently support the use of data of a long time span to test for long-run PPP.

4. CONCLUDING REMARKS

A generalized notion of cointegration, called fractional cointegration, has been introduced and used to examine the empirical relevance of long-run PPP based on historical data for the 1914–1989 period. The analysis of fraction cointegration allows the equilibrium error to follow a fractionally integrated process and avoids the stringent I(1) and I(0) distinction maintained in previous empirical work. By allowing for fractionally integrated equilibrium errors, it provides a flexible and parsimonious way to model the low-frequency dynamics of deviations from equilibrium. Accordingly, the fractional cointegration analysis can identify a wide range of mean-reversion behavior, which turns out to be important for a proper evaluation of long-run PPP. The empirical results show that the evidence from the fractional cointegration analysis is much more favorable to long-run PPP than that from the standard cointegration analysis using unit-root tests and that in three out of five PPP relationships investigated the equilibrium error can be characterized by a fractionally integrated process.

The findings of the fractionally integrated property of PPP deviations for many different countries invite questions concerning the source of such property. Johansen and Juselius (1990b) recently suggested that deviations from PPP for the United Kingdom can be accounted for by interactions between exchange rates and interest rates. It is possible that the behavior of PPP deviations may in general reflect the statistical property of economic fundamentals such as the levels of output, money supply, and interest rates. In this regard, the studies by Diebold and Rudebusch (1989), Porter-Hudak (1990), and Shea (1991) are particularly interesting. These studies individually report evidence of fractional integration in output, the money supply, and interest rates. Of course, more systematic studies appear required to determine the source of deviations from PPP, and this is a potential topic for future research.

Another line of future research is on the estimation and testing of fractionally cointegrated systems. Little research has sought to analyze fractionally cointegrated systems. This article takes a step in that direction. It is

Table 6. Results of the GPH Test for Cointegration

	$\mu = .55$			$\mu = .575$			$\mu = .60$		
Country	d	H_0: $d = 1$	H_1: $d = 0$	d	H_0: $d = 1$	H_1: $d = 0$	d	H_0: $d = 1$	H_1: $d = 0$
Canada	.504	−1.679*	1.709**	.558	−1.602*	2.021**	.675	−1.320	2.739**
France	.473	−1.784*	1.604*	.549	−1.633*	1.991**	.540	−1.867*	2.193**
Italy	.352	−2.197**	1.192	.346	−2.405**	1.218	.464	−2.178**	1.882**
Japan	.572	−1.452	1.937**	.535	−1.685*	1.939**	.519	−1.951*	2.108**
U.K.	.509	−1.664*	1.724**	.598	−1.548*	2.165**	.594	−1.647*	2.412**

NOTE: The sample size for the GPH test is given by $n = T^{\mu}$. Significance is indicated by * at the 10% level or ** at the 5% level. The hypothesis H_0: $d = 1$ is tested against the one-sided alternative of $d < 1$. The hypothesis H_0: $d = 0$ is tested against the two-sided alternative of $d \neq 0$. Critical values are based on simulated values given in Table 1.

shown that the Engle–Granger (1987) least squares approach can still be applied to estimate long-run equilibrium relationships under fractional cointegration. This least squares approach is presumably not as efficient as a full-system-estimation approach. Johansen (1991) and Phillips (1991), for example, suggested the use of full-system estimation of the error-correction model under nonfractional cointegration. It is certainly of interest to develop an optimal method for estimating fractional error-correction models. As another interesting approach, Sowell (1988) explored full maximum likelihood estimation of fractionally cointegrated systems. By extending Li and McLeod's (1986) univariate analysis, Sowell (1988) derived the likelihood function of multivariate fractional processes under the assumptions of normal innovations and known process means. Although the full maximum likelihood estimation technique can be statistically powerful, the maintained assumptions of normality and known means appear not appropriate in the empirical application here. Furthermore, implementation of Sowell's (1988) multivariate maximum likelihood technique is not straightforward in terms of hypothesis testing. In constructing the multivariate likelihood function, the cointegrating vector is not identifiable under the null hypothesis of no cointegration. As Sowell (1988) observed, this leads to a nonstandard hypothesis testing problem such that one may obtain only the upper or lower bound but not the exact distribution of the test statistic.

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APPENDIX: A SIMULATION EXPERIMENT FOR POWER COMPARISON

To illustrate the potential difference in power between the GPH test and the ADF test for cointegration, a Monte Carlo experiment, similar to that of Engle and Granger (1987), is conducted. A bivariate system of x_{1t} and x_{2t} is modeled by

$$x_{1t} + x_{2t} = u_{1t} \tag{A.1}$$

and

$$x_{1t} + 2x_{2t} = u_{2t}, \tag{A.2}$$

with $(1 - L)u_{1t} = \varepsilon_{1t}$, and u_{2t} is generated alternatively as an autoregressive process

$$(1 - \phi L)u_{2t} = \varepsilon_{2t} \tag{A.3}$$

or a fractional white-noise process

$$(1 - L)^d u_{2t} = \varepsilon_{2t}. \tag{A.4}$$

The innovations ε_{1t} and ε_{2t} are generated as independent standard normal variates. When u_{2t} is given by (A.3) and $|\phi| < 1$, x_{1t} and x_{2t} are cointegrated and (A.2) is their cointegrating relationship. Alternatively, when u_{2t} is given by (A.4) and $d < 1$, x_{1t} and x_{2t} are (fractionally) cointegrated. When either $\phi = 1$ in (A.3) or $d = 1$ in (A.4), however, the two series are not cointegrated. In the experiment, samples of size $T = 76$ are used. Sample series of x_1 and x_2 are generated by setting the initial values of u_1 and u_2 equal to 0 and creating 126 observations, of which the first 50 observations are discarded to reduce the effect of the initial condition. For each pair of series generated, the ADF and GPH tests are employed to test for cointegration, and cases of rejection are recorded for the individual tests. Levels of significance used are 5% and 10%, and results are based on 10,000 replications.

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